SECTION 14.4: THE LENGTH OF CURVES

RECALL: To find the length of a curve C described parametrically: $\{x = f(t),\ y = g(t)$, for $a \le t \le b$:

$$s = \int_C ds = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

The quantity $ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$ is called the **arc length differential**.

DEFINITION: If a curve C is traced out by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $a \le t \le b$ where $\vec{r}$ is smooth,

$$s = \int_C ds = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \|\vec{r}\,'(t)\|\, dt = \int_a^b \|\vec{v}(t)\|\, dt$$

Here, $ds = \|\vec{v}(t)\|\, dt$, the arc length differential, may be interpreted as: $\dfrac{ds}{dt} = \|\vec{v}\|$. Does this make sense?

EXAMPLE 1: Let C be the helix traced out by the v.v.f. $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t\rangle$ for $t \ge 0$.

1. Find the length of the curve between the points $(1, 0, 0)$ and $(-1, 0, 12\pi)$.

Ans: 13π units

2. Find and interpret: $s(t) = \displaystyle\int_0^t \|\vec{v}(u)\|\, du$

Ans: $s(t) = 13t$ units gives the distance traveled as of time t. (an 'odometer' function.)

THE ARC LENGTH PARAMETER: If $\vec{r}$ is a smooth v.v.f. tracing out a curve C for $a \le t \le b$,

$$s(t) = \int_a^t \|\vec{r}\,'(u)\|\, du$$

the called the **arc length parameter** and gives how far along a path one has traveled at time t.

EXAMPLE 2: For $\vec{r}(t) = \langle \cos(5t), \sin(5t), 12t \rangle$, $t \ge 0$, we have $s(t) = 13t$ or $s = 13t$.

1. Solve for t in terms of s and re-parametrize $\vec{r}$ as a function of s, $\vec{r}(s)$.

Ans: $\vec{r}(s) = \left\langle \cos\left(\frac{5s}{13}\right), \sin\left(\frac{5s}{13}\right), \frac{12s}{13} \right\rangle$

2. Find $\vec{v}(s) = \frac{d\vec{r}}{ds}$ and $\|\vec{v}(s)\| = \left\|\frac{d\vec{r}}{ds}\right\|$. Are you surprised? Think about what $\left\|\frac{d\vec{r}}{ds}\right\|$ measures.

Ans: $\vec{v}(s) = \left\langle -\frac{5}{13}\sin\left(\frac{5s}{13}\right), \frac{5}{13}\cos\left(\frac{5s}{13}\right), \frac{12}{13} \right\rangle$ and $\|\vec{v}(s)\| = 1$.

THEOREM: If $\vec{r}$ is smooth, then $\|\vec{v}(t)\| = \left\|\frac{d\vec{r}}{dt}\right\| = 1$ if, and only if, $t = s$, the arc length parameter.

EXAMPLE 3: Recall circular motion of radius $R > 0$ with angular frequency $\omega > 0$ can be described by:

$$\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t) \rangle$$

Re-parametrize $\vec{r}$ using the arc length parameter and verify $\|\vec{v}(s)\| = 1$.

Ans: $\vec{r}(s) = \left\langle R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right) \right\rangle$.

NOTE: The arc length parameter is one of those things that is nice to use to define and discuss concepts theoretically, but, in practice, we avoid using it due to the (usual) complexity of its calculation.

HOMEWORK: Section 14.4: 9 - 53 every other odd.